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LETTER TO THE EDITOR

Wetting velocity near the directed percolation threshold

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Abstract. Near the directed percolation threshold, p_d, the wetting velocity v scales as $(1-v) \sim (p_d - p)^{\theta'}$. We extrapolate from calculations on strips of finite width to show that, on the two-dimensional square lattice, $\theta' = 1.73 \pm 0.02$. This agrees with a scaling argument by Barma and Ray which predicts $\theta' = \nu_{\parallel}$.

In recent years numerical work on strips of finite width has given considerable insight into the properties of spin and percolative systems (Vannimenus and Nadal 1984). In this letter we present results for a geometric problem defined on a percolation cluster, the wetting velocity.

We consider a square lattice where bonds are present with probability p and absent with probability $1-p$ (Stauffer 1985). For $p \geqslant p_c$, the critical percolation threshold, an infinite number of connected bonds span the lattice. Let (x_1, y_1) and (x_2, y_2) be two points on the infinite cluster with cartesian separation

$$L = |x_1 - x_2| + |y_1 - y_2|. \tag{1}$$

The chemical distance, L_{chem}, between the points is defined to be the shortest path along occupied bonds linking the two points (Havlin and Nossal 1984). The wetting velocity, v, is then given by (Dhar 1982, Barma 1985, Barma and Ray 1986)

$$v = \lim_{L \to \infty} \frac{L}{L_{\text{chem}}}. \tag{2}$$

The variation of v with p is interesting (Dhar 1982, Barma 1985) because it exhibits singularities not only at the percolation threshold, p_c, where

$$v \sim (p - p_c)^{\theta} \qquad p \to p_c^+ \tag{3}$$

but also at the directed percolation threshold (Kinzel 1983), $p_d(\alpha)$,

$$1 - v \sim (p_d(\alpha) - p)^{\theta'(\alpha)} \qquad p \to p_d^- \tag{4}$$

where α is the angle between the direction along which v is measured and a diagonal of the square lattice.

Extensive numerical simulations (Havlin and Nossal 1984, Herrmann *et al* 1984, Vannimenus *et al* 1984, Barma 1985, Edwards and Kerstein 1985, Grassberger 1985, Havlin *et al* 1985, Martin 1985) have been carried out to determine the behaviour of v near p_c. In these the exponent $\tilde{\nu}$ defining the scaling of L with L_{chem} ($L^2 \sim \langle L_{\text{chem}} \rangle^{2\tilde{\nu}}$) is determined. Using scaling arguments, it is easy (Barma 1985) to show that

$$\theta \sim \nu(1/\tilde{\nu} - 1) \tag{5}$$

where ν is the correlation length exponent. In two dimensions $\tilde{\nu} = 0.883 \pm 0.003$ and $\nu = \frac{4}{3}$. Therefore $\theta = 0.176 \pm 0.005$.

The behaviour of v near p_d has, however, remained largely unexplored. In this letter we consider the behaviour of the wetting velocity between two points lying along the diagonal of a square lattice as $p \to p_d$. The geometry which is relevant to this problem is shown in figure 1. We shall describe the diagonal between the allowed axes as the easy direction and use α to describe an angle relative to this direction. In directed percolation (Kinzel 1983) flow occurs along bonds for which $\alpha = \pm\pi/4$. p_d depends on α (Domany and Kinzel 1981) and hence the wetting velocity will become unity at a value of $p = p_d$ which is α dependent. Moreover, the associated exponent will depend on α. In this letter we restrict ourselves to the case $\alpha = 0$ for which p_d is accurately known.

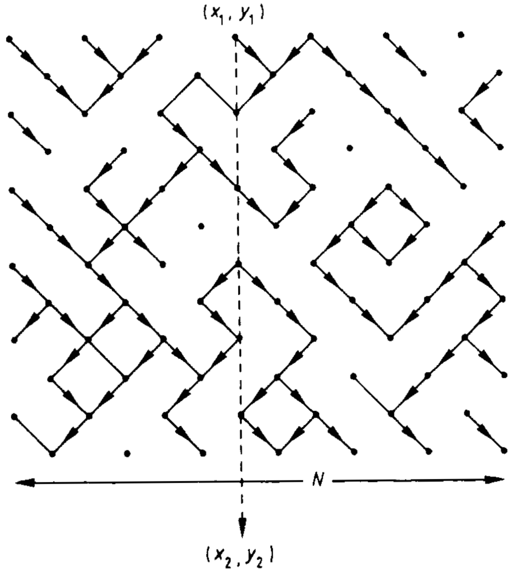

Figure 1. A strip of width N. The arrows show the allowed flow if the directed problem is considered. Points (x_1, y_1) and (x_2, y_2) lie along the diagonal $\alpha = 0$.

An analytic solution of the problem on a Cayley tree gives $\theta' = 1$ with logarithmic corrections (Barma 1985) and real space renormalisation group results have been obtained for a hierarchical lattice (Barma and Ray 1986). Both results are consistent with a scaling argument due to Barma and Ray (1986) which suggests $\theta' = \nu_\parallel$, where $\nu_\parallel$ is the parallel correlation length exponent for directed percolation (Kinzel and Yeomans 1981). We summarise this argument and then show that for the two-dimensional square lattice strip calculations indicate that $\theta' = \nu_\parallel$ to within the accuracy of our results.

Consider two points along the diagonal ($\alpha = 0$) of a connected cluster. Let the cartesian separation between the two points be $L \gg \xi_\parallel$, where $\xi_\parallel$ is the parallel correlation length in directed percolation (Kinzel 1983). Close to p_d, one would expect that the shortest path length between the two points will be made up of many directed segments of length $\xi_\parallel$ with the segments connected by bridges, which include bonds traversed antiparallel to the easy directions, of mean length λ. v is then given by

$$v \underset{p\to p_d^-}{\sim} \frac{\xi_\parallel}{\xi_\parallel + \lambda}. \tag{6}$$

Re-writing this to demonstrate the divergence of $\xi_\parallel$ at p_d, we obtain

$$1 - v \sim \lambda (p_d - p)^{\nu_\parallel} \qquad p \to p_d^-. \tag{7}$$

Therefore, if λ does not diverge at p_d, $\theta' = \nu_\parallel$.

To obtain the value θ' for the two-dimensional square lattice, a strip of width N with periodic boundary conditions in the transverse direction was built up row by row for bond concentration $p = p_d = 0.6445$ (Kinzel and Yeomans 1981) using a random number generator. The strip was oriented along $\alpha = 0$. As each row was added the shortest path was determined at every point, care being taken to include connections from paths looping backwards.

A problem which arises immediately is that the strip is broken with probability $(1-p)^N$. When this occurred the values of L and L_{chem} were recorded and the counting recommenced. Between 2000 and 3000 trials were made for widths up to $N = 12$ and a lesser number of trials for N larger than 12. This was adequate since the fluctuations from one sample to another were smaller for large strip widths because the typical length of the strip generated was around 20 000. Values of (L, L_{chem}) were then plotted as shown in figure 2 for $N = 13$. The points lay on a good straight line for all values of N and for $L \geqslant 100$. Therefore, if the strip was of length $L < 100$, the corresponding results were ignored. The remaining points were fitted to a straight line of which the gradient $1/v$ gave the wetting velocities listed for values of N from 9 to 15 in table 1. For lower values of N (between 4 and 8), it was difficult to generate many samples of length $L \geqslant 100$ and this introduced large errors in the estimation of $v(N)$. For $N > 15$, the computer time needed to achieve sufficient accuracy was prohibitively large.

Finite-size scaling (Vannimenus and Nadal 1984) implies that

$$1 - v(N) \sim N^{-\theta'/\nu_\perp} \tag{8}$$

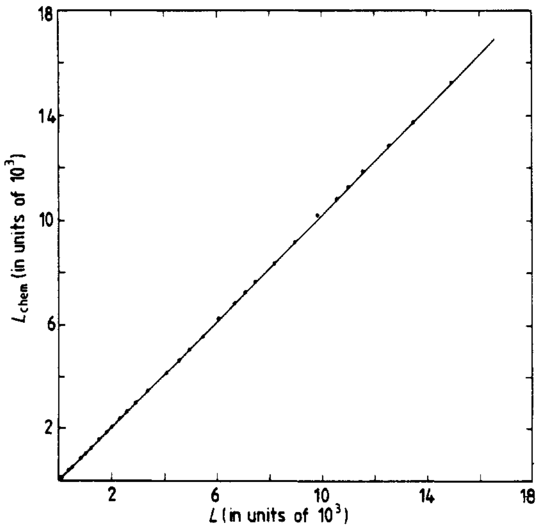

Figure 2. Variation of the shortest path length L_{chem} with the cartesian separation L for a typical value of the strip width, $N = 13$.

Table 1. The wetting velocity for different strip widths. The figure in the bracket denotes the error in the last digit.

N	v
9	0.967 (3)
10	0.971 (2)
11	0.975 (1)
12	0.978 (1)
13	0.981 (1)
14	0.9827 (6)
15	0.9847 (5)

at p_d. Note that $\nu_\perp$, the transverse correlation length exponent of the directed percolation problem (Kinzel 1983), and not $\nu_\|$ appears in this expression because it is this correlation length that scales with the width of the strip (Kinzel and Yeomans 1981). $\ln(1-v)$ was plotted against $\ln N$ as shown in figure 3. Using the value $\nu_\perp = 1.103$ (Kinzel and Yeomans 1981) then gives

$$\theta' = 1.73 \pm 0.02 \tag{9}$$

for the square lattice. For directed percolation on the square lattice $\nu_\| = 1.739 \pm 0.002$ (Kinzel and Yeomans 1981) and hence the numerical evidence supports the scaling argument of Barma and Ray (1986). Note that the argument depends on there being a finite number of steps in any backward loop at p_d. Our simulations gave no evidence that this is untrue. Further work on the Cayley tree also supports this statement (Barma and Ramaswamy 1986).

We close by noting that the wetting velocity problem maps onto the behaviour of a random network of non-linear resistors with $I-V$ characteristic

$$V = |I|^\alpha R \operatorname{sgn} I \tag{10}$$

in the limit $\alpha \to 0$ (Blumenfeld and Aharony 1985, de Arcangelis *et al* 1985).

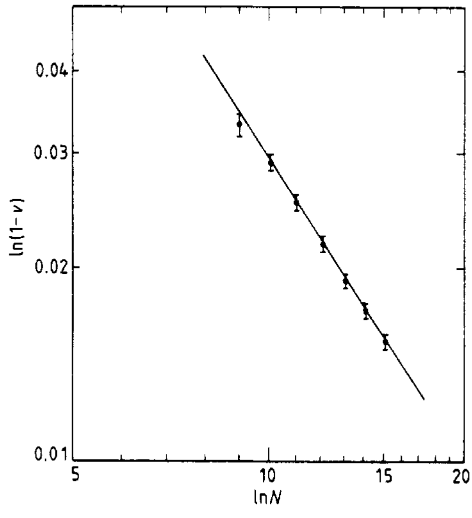

Figure 3. Variation of $\ln(1-v)$ with $\ln(N)$ at $p = 0.6445$ for strip widths $N = 9$ to 15.

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References

Barma M 1985 *J. Phys. A: Math. Gen.* **18** L277
Barma M and Ramaswamy R 1986 *J. Phys. A: Math. Gen.* **19** L605
Barma M and Ray P 1986 *Phys. Rev.* B **34** 3403
Blumenfeld R and Aharony A 1985 *J. Phys. A: Math. Gen.* **18** L443
de Arcangelis L, Coniglio A and Redner S 1985 *J. Phys. A: Math. Gen.* **18** L805
Dhar D 1982 *J. Phys. A: Math. Gen.* **15** 1859
Domany E and Kinzel W 1981 *Phys. Rev. Lett.* **47** 5
Edwards B F and Kerstein A R 1985 *J. Phys. A: Math. Gen.* **18** L1081
Grassberger P 1985 *J. Phys. A: Math. Gen.* **18** L215
Havlin S and Nossal R 1984 *J. Phys. A: Math. Gen.* **17** L427
Havlin S, Trus B, Weiss G H and Ben-Avraham D 1985 *J. Phys. A: Math. Gen.* **18** L247
Herrmann H J, Hong D C and Stanley H E 1984 *J. Phys. A: Math. Gen.* **17** L261
Kinzel W 1983 *Percolation Structures and Processes* ed G Deutscher, R Zallen and J Adler (Bristol: Adam Hilger) p 425
Kinzel W and Yeomans J M 1981 *J. Phys. A: Math. Gen.* **14** L163
Martin J E 1985 *J. Phys. A: Math. Gen.* **18** L207
Stauffer D 1985 *Introduction to Percolation Theory* (London: Taylor and Francis)
Vannimenus J and Nadal J P 1984 *Phys. Rep.* **103** 47
Vannimenus J, Nadal J P and Martin H 1984 *J. Phys. A: Math. Gen.* 17 L351